EJ

Simplifying the optimal win-rate against a fixed loss rebate & game.

Recall from LRT1:

$$p = p(x,b) = \frac{e^{\frac{-2\mu x}{\sigma^2}} - 1}{e^{\frac{-2\mu b}{\sigma^2}} - 1}$$

Let $C = \frac{-2\mu}{\sigma^2}$, $-\frac{1}{C} = \frac{\sigma^2}{2\mu}$

$$p = \frac{e^{cx} - 1}{e^{cb} - 1} \quad (1)$$

$$w = w(x,b,L) = (1-L)(1-p)(-x) + p(b-x)$$

$$w = -x + \cancel{px} + \cancel{Lp}^{\,Lx} - Lpx + pb - \cancel{px}$$

$$w = -x + \cancel{Lp}^{\,Lx} - Lpx + pb \quad (2)$$

The values of x, b that maximize w are:

$$\begin{cases} x = -\frac{1}{c}\left(1 + \frac{\ln(1-L)}{L}\right) \\ b = -\frac{1}{c}\ln(1-L) \end{cases} \quad (3)$$

Plug (3) into (1) + simplify

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Numerator of (1)

$$e^{cx} \not{A} = e^{-(1+\frac{\ln(1-L)}{L})}$$

$$= e^{-1} \cdot e^{-\frac{\ln(1-L)}{L}}$$

$$= \frac{1}{e} \cdot \left(e^{\ln(1-L)}\right)^{-\frac{1}{L}}$$

$$= \frac{1}{e} (1-L)^{-\frac{1}{L}} = \frac{1}{e(1-L)^{\frac{1}{L}}}$$

Let $M = (1-L)^{\frac{1}{L}}$. Then

$$\boxed{e^{cx} - 1 = \frac{1}{eM} - 1}$$

Denominator of (1)

$$e^{cb} = e^{-\ln(1-L)} = \frac{1}{1-L}$$

So $e^{cb} - 1 = \frac{1}{1-L} - 1 = \frac{1-(1-L)}{1-L}$

$$\boxed{e^{cb} - 1 = \frac{L}{1-L}}$$

So $p = \left(\frac{1-L}{L}\right) \cdot \left(\frac{1}{eM} - 1\right)$ (4)

$$= \left(\frac{1}{L} - 1\right)\left(\frac{1}{eM} - 1\right) = \frac{1}{eLM} - \frac{1}{L} - \frac{1}{eM} + 1$$

Let $R = \ln(1-L)$. Again, I am just trying to simplify. Eventually c, M, R will go back to their original form.

Recall $\omega = -X + LX - LpX + pb$

(5) $$-X = \frac{1}{c}\left(1 + \frac{\ln(1-L)}{L}\right) = \frac{1}{c}\left(1 + \frac{R}{L}\right) = \frac{1}{c} + \frac{R}{Lc}$$ (5)

(6) $$LX = -\frac{L}{c} - \frac{R}{c}$$ (6)

$$-LpX = LX(-p) = \left(-\frac{L}{c} - \frac{R}{c}\right)\left(\frac{-1}{eLM} + \frac{1}{L} + \frac{1}{eM} - 1\right)$$

(7) $$-LpX = \frac{1}{eMc} - \frac{1}{c} - \frac{L}{eMc} + \frac{L}{c} + \frac{R}{eLMc} - \frac{R}{Lc} - \frac{R}{eMc} + \frac{R}{c}$$

$$pb = \left(\frac{1}{eLM} - \frac{1}{L} - \frac{1}{eM} + 1\right)\left(-\frac{R}{c}\right)$$

(8) $$pb = \frac{-R}{eLMc} + \frac{R}{Lc} + \frac{R}{eMc} - \frac{R}{c}$$

We add (5) + (6) + (7) + (8) and simplify

$$\omega = \cancel{\frac{V}{c}} + \frac{R}{Lc} - \cancel{\frac{V}{c}} - \cancel{\frac{R}{c}}$$

$$+ \frac{1}{eMc} - \cancel{\frac{V}{c}} - \frac{L}{eMc} + \cancel{\frac{V}{c}} + \cancel{\frac{R}{eLMc}} - \cancel{\frac{R}{Lc}} - \cancel{\frac{R}{eMc}} + \cancel{\frac{R}{c}}$$

$$- \cancel{\frac{R}{eLMc}} + \cancel{\frac{R}{Lc}} + \cancel{\frac{R}{eMc}} - \frac{R}{c}$$

(9) $$\omega = \frac{R}{Lc} + \frac{1}{eMc} - \frac{L}{eMc} - \frac{R}{c}$$ simplifies a lot!

$$\omega = \left(\frac{1}{c}\right)\left(\frac{R}{L} + \frac{1}{eM} - \frac{L}{eM} - \frac{R}{\cancel{c}}\right)$$

$$\omega = \left(\frac{1}{c}\right)\left(\frac{R}{L} + \frac{1-L}{eM} - \frac{R}{\cancel{c}}\right)$$

$$= \left(\frac{1}{c}\right)\left(\frac{\ln(1-L)}{L} + \frac{1-L}{e\cdot(1-L)^{\frac{1}{L}}} - \ln(1-L)\right)$$

$$= \cancel{\frac{\sigma^2}{2\mu}} \left(\frac{1}{c}\right)\cdot\left(\ln(1-L)\cdot\left(\frac{1}{L}-1\right) + \frac{1-L}{e\cdot(1-L)^{\frac{1}{L}}}\right)$$

$$= \left(\frac{1}{c}\right)\left(\ln(1-L)\left(\frac{1-L}{L}\right) + \frac{1-L}{e(1-L)^{\frac{1}{L}}}\right)$$

$$= \left(\frac{1}{c}\right)\left(\frac{1-L}{L}\right)\cdot\left(\ln(1-L) + \frac{L}{e\cdot(1-L)^{\frac{1}{L}}}\right)$$

$$= \left(\frac{\sigma^2}{2\mu}\right)\cdot\left(\frac{L-1}{L}\right)\cdot\left(\ln(1-L) + \frac{L}{e(1-L)^{\frac{1}{L}}}\right)$$

(10) $$\omega = \left(\frac{-\sigma^2}{2\mu}\right)\cdot\left(\frac{1-L}{L}\right)\cdot\left(\ln(1-L) + \frac{L}{e(1-L)^{\frac{1}{L}}}\right)$$

Define the loss rebate coefficient to be:

(11) $$f(L) = \left(\frac{1-L}{L}\right)\cdot\left(\ln(1-L) + \frac{L}{e(1-L)^{\frac{1}{L}}}\right)$$

Then

(12) $$\omega = \omega(\mu, \sigma, L) = \left(\frac{-\sigma^2}{2\mu}\right)\cdot f(L)$$

Theorem. The maximum win amount exploiting a loss rebate on a Game G with loss rebate L is

$$\text{maxwin}(\mu, \sigma, L) = \left(\frac{-\sigma^2}{2\mu}\right)\cdot f(L)$$